CRM-3326 (2013)

Symmetric polynomials, generalized Jacobi-Trudi identities and τ-functions*

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Abstract

An element $[\Phi] \in Gr_n(\mathcal{H}_+, \mathbf{F})$ of the Grassmannian of n-dimensional subspaces of the Hardy space $\mathcal{H}_+ = H^2$, extended over the field $\mathbf{F} = \mathbf{C}(x_1, \dots, x_n)$, may be associated to any polynomial basis $\phi = \{\phi_0, \phi_1, \cdots\}$ for $\mathbf{C}(x)$. The Plücker coordinates $S^{\phi}_{\lambda,n}(x_1, \dots, x_n)$ of $[\Phi]$, labelled by partitions λ, provide an analog of Jacobi's bi-alternant formula, defining a generalization of Schur polynomials. Applying the recursion relations satisfied by the polynomial system ϕ to the analog $\{h^{(0)}_i\}$ of the complete symmetric functions generates a doubly infinite matrix $h^{(j)}_i$ of symmetric polynomials that determine an element $[H] \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$. This is shown to coincide with $[\Phi]$, implying a set of generalized Jacobi identities, extending a result obtained by Sergeev and Veselov [25] for the case of orthogonal polynomials. The symmetric polynomials $S^{\phi}_{\lambda,n}(x_1, \dots, x_n)$ are shown to be KP (Kadomtsev-Petviashvili) τ-functions in terms of the monomial sums $[x]$ of the x_a's, viewed as KP flow variables. A fermionic operator representation is derived for these, as well as for the infinite sums $\sum_\lambda S^{\phi}_{\lambda,n}([x]) S^{\theta}_{\lambda,n}(\mathbf{t})$ associated to any pair of polynomial bases (ϕ, θ), which are shown to be 2D Toda lattice τ-functions. A number of applications are given, including classical group character expansions, matrix model partition functions and generators for random processes.

*Work of J.H. supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds Québecois de la recherche sur la nature et les technologies (FQRNT).

1 Introduction

The Jacobi-Trudi identities [20] express Schur polynomials $S_\lambda(x_1, x_2, \dots, x_n)$, labelled by partitions λ of length $\ell(\lambda) \leq n$, as determinants

$$S_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j}) \tag{1.1}$$

in terms of the complete and elementary symmetric functions

$$h_j := S_{(j)}, \quad e_j := S_{(1^j)}. \tag{1.2}$$

Here $\lambda = (\lambda_1 \geq \lambda_2, \cdots \geq \lambda_{\ell(\lambda)} > 0)$ denotes a partition and λ' the conjugate partition, whose Young diagram is the transpose of the one for λ.

In recent years, a number of examples have been found in which a generalized form of such identities are satisfied by other classes of functions [21, 7, 2, 17, 5, 25], also labelled by partitions, together with an additional integer parameter. These are referred to as generalized, or "quantum" Jacobi-Trudi identities. In particular, they are known to be satisfied by the eigenvalues of commuting transfer matrices in the R-matrix approach to quantum integrable systems, due to the underlying Yangian algebra structure, and are known in this case as the Bazhanov-Cherednik-Reshetikhin formula [7, 2, 5].

Another recently studied case [25] involves a generalization $S^\phi_{\lambda,n}$ of Schur functions obtained by replacing the monomials appearing in the bi-alternant formula for S_λ by a sequence of orthogonal polynomials $\{\phi_i\}_{i\ =0,1,\dots}$. In this case, the additional integer labels a sequence of functions $\{h_i^{(j)}\}_{i,j\in\mathbf{N}}$ defined recursively in terms of the analogs $h_i^{(0)} := S^\phi_{(i),n}$ of the complete symmetric functions. More generally, it has been shown [5], that such generalized Jacobi-Trudi identities are satisfied by the coefficients of the expansion of any sequence of KP τ-functions determining a solution of the MKP integrable hierarchy in a basis of Schur functions.

In the following, we extend the results of ref. [25] to the case of arbitrary polynomials bases $\phi := \{\phi_i\}_{i\ =0,1,\dots}$ of $\mathbf{C}(x)$, using a geometrical approach that associates to any such polynomial system a corresponding element $[\Phi]$ of an infinite Grassmann manifold, whose Plücker coordinates coincide with the functions $S^\phi_{\lambda,n}$. The meaning of the Jacobi-Trudi identities becomes clear in this setting; they follow from the fact that the element $[H]$ of the Grassmannian determined from applying the same set of recursions relations as those satisfied by the polynomials $\{\phi_i\}_{i\ =0,1,\dots}$ to the analogs $\{h_i^{(j)}\}_{i,j\in\mathbf{N}}$ of the complete symmetric functions is, within a change of basis, identical to the element $[\Phi]$ determined by evaluation of the polynomials themselves at the various parameter values $\{x_1, \dots, x_n\}$,

and hence their Plücker coordinates coincide. This result is the content of Proposition 2.1, Section 2.2, with a dual version given in Section 2.3. Two methods of proof are given: induction in the successive basis elements defining $[H]$, and the "dressing" method, based on transforming the standard case, involving monomials systems, to the general one.

In Section 3, it is shown that the resulting functions $S^{\phi}_{\lambda,n}([x])$ may be viewed as KP τ-functions in terms of flow variables identified as monomial sums $[x]$ over the parameters $\{x_1, \dots, x_n\}$. Moreover, since the symmetric polynomials $S^{\phi}_{\lambda,n}([x]_)$ also satisfy the Plücker relations, they may be used as coefficients in a Schur function series defining a parametric family of KP τ-functions

$$\tau_\phi(n, [x], \mathbf{t}) = \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} S^{\phi}_{\lambda,n}(x_1, \dots, x_n) S_\lambda(\mathbf{t}), \tag{1.3}$$

where the variables $\mathbf{t} = (t_1, t_2, \dots)$ are viewed as additional KP flow parameters. A fermionic representation of these τ-functions is given, and used to show that they form a lattice of τ-functions in the $\mathbf{t}$ variables which, when combined with the dependence on the monomials sum variables $[x]$, may also be viewed as a 2D-Toda τ-functions. More generally, summing the products of generalized Schur functions $S^{\phi}_{\lambda,n}$ and $S^{\theta}_{\lambda,n}$ corresponding to a pair (ϕ, θ) of such polynomials systems

$$\tau_{\phi,\theta}(n, \mathbf{s}, \mathbf{t}) = \sum_{\lambda} S^{\phi}_{\lambda,n}(\mathbf{s}) S^{\theta}_{\lambda,n}(\mathbf{t}) \tag{1.4}$$

provides a broader class of 2D-Toda τ-functions.

In Section 4, these results are iapplied to various examples, including: determinantal representations of irreducible characters of the classical Lie groups and their Schur function expansions; generalized matrix model partition functions and generating functions for certain exclusion processes.

2 Polynomial systems and Grassmannians

2.1 Generalized Schur polynomials S^{ϕ}_{λ}

Let $\mathbf{F} := \mathbf{C}(x_1, \dots, x_n)$ denote the extension of the field of complex numbers by the indeterminates $\{x_1, x_2, \dots, x_n\}$. Let $\mathcal{H}_+$ denote the space of those square integrable complex functions on the unit circle $|z| = 1$ that admit a holomorphic extension to the interior disc (i.e., the Hardy space H^2). We use the monomial basis $\{\mathbf{b}_i := z^{i-1}\}_{i\in\mathbf{N}^+}$ to represent

elements as semi-infinite column vectors, labelled increasingly from the bottom element upward; i.e. , $\mathbf{b}_1 \sim (\dots, 0, 0, 1)^t$, $\mathbf{b}_2 \sim (\dots, 0, 1, 0)^t$, etc.

Denote by $\mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ the Grassmannian of n-dimensional subspaces of

$$\mathcal{H}_+(\mathbf{F}) := \mathbf{F} \otimes_{\mathbf{C}} \mathcal{H}_+. \tag{2.1}$$

Relative to the basis $\{\mathbf{b}_i\}$, an element $[W] \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ may be represented by a semi-infinite $(\infty \times n)$ rank n matrix W whose columns $(W^1, \dots, W^n)$ span $[W]$. Two such rank-n matrices related by $W = \tilde{W} g$, $g \in Gl(n, \mathbf{F})$, span the same subspace, and hence belong to the same equivalence class $[W] = [Wg]$. The n-component row vectors of W, corresponding to the components along the basis elements b_i will be denoted

$$W_i := (W_{i1}, \quad \dots W_{in}), \quad i \in \mathbf{N}. \tag{2.2}$$

For any integer partition D

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \ \lambda_{\ell(\lambda)} \geq 0), \quad \lambda_i \in \mathbf{N}^+ \tag{2.3}$$

of weight

$$|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i \tag{2.4}$$

and length $\ell(\lambda)$, we define an infinite, strictly decreasing sequence of integers $l_1 > l_2 > \dots$ (sometimes called "particle coordinates") by

$$l_i := \lambda_i - i + n, \quad i \in \mathbf{N}^+, \tag{2.5}$$

with the convention that $\lambda_i := 0$ for $i > \ell(\lambda)$. After $\ell(\lambda)$ terms, these become the successive decreasing integers $(n - \ell(\lambda), n - \ell(\lambda) - 1, \dots)$. For any partition of length

$$\ell(\lambda) \leq n \tag{2.6}$$

let W_λ denote the $n \times n$ minor of W whose ith row (counting from the top down) is W_{l_i+1}. Then the λth Plücker coordinate of the image of the element W under the Plücker map

$$\mathfrak{Pl} : \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F}) \rightarrow \bigwedge^n \mathcal{H}_+(\mathbf{F}) \tag{2.7}$$

is given, within projective equivalence, by the determinant

$$\pi_\lambda(W) := \det(W_\lambda). \tag{2.8}$$

These satisfy the Plücker relations which, on the "big cell" (i.e. where $\pi_0(W) \neq 0$), are equivalent to the generalized Giambelli identity (see e.g, [11], Corollary 2.1):

$$\pi_0(W)^{r-1}\pi_\lambda(W) = \det(\pi_{(a_i|b_j)})|_{1\leq i,j\leq r}. \tag{2.9}$$

Here $(a_1, \ldots, a_k|b_1, b_2, \ldots b_r)$ is Frobenius' notation for the partition λ, with (a_i, b_i) the arm and leg lengths of the Young diagram for λ, measured from the ith diagonal, $(a_i|b_j)$ denotes the hook partition $\lambda = (1 + a_i, 1^{b_j})$ and r is the Frobenius rank of λ (i.e., the length of the central diagonal).

Given a basis for $\mathbf{C}[x]$ consisting of monic polynomials $\{\phi_i(x),\ \deg\phi_i = i\}_{i\in\mathbf{N}}$, we associate an $\infty \times n$ matrix Φ with components

$$\Phi_{ij} := \phi_{i-1}(x_j) \tag{2.10}$$

that determines an element $[\Phi] \in \mathrm{Gr}_n(\mathcal{H}^n_+, \mathbf{F})$. The ith row vector is thus

$$\Phi^t_i := (\phi_{i-1}(x_1), \ldots, \phi_{i-1}(x_n)). \tag{2.11}$$

Let Λ denote the semi-infinite upper triangular shift matrix (with 1's above the principal diagonal and zeros elsewhere), representing multiplication by z:

$$z : \mathbf{b}_i \rightarrow \mathbf{b}_{i+1}. \tag{2.12}$$

and Λ^t its (lower triangular) transpose, and define

$$X := \mathrm{diag}(x_1, \ldots x_n). \tag{2.13}$$

It follows that there is a unique semi-infinite upper triangular recursion matrix J^+ such that

$$\Phi X = J\Phi \tag{2.14}$$

where

$$J := \Lambda^t + J^+. \tag{2.15}$$

(In the special case where the ϕ_i's form an orthonormal system with respect to some inner product, the matrix J is tridiagonal, but this will not be assumed here.)

Define the infinite triangular matrix A^ϕ, with 1's along the diagonal, whose rows are the coefficients of the polynomials $\{\phi_i\}$,

$$A^\phi_{ij} := \phi_{i-1,j-1} \quad \text{if} \quad i \geq j, \quad A^\phi_{ij} = 0 \quad \text{if} \quad i < j, \quad i, j \in \mathbf{N}^+, \tag{2.16}$$

where

$$\phi_i(x) = \sum_{j=0}^{i} \phi_{i,j} x^j. \tag{2.17}$$

(Note that in our notational conventions this is *upper* triangular, because we count upward from the bottom, starting with 1.) For the case of monomials, denote the matrix Φ as $\overset{0}{\Phi}$. We then have

$$\Phi = A^{\phi}\overset{0}{\Phi}. \tag{2.18}$$

Since $\overset{0}{\Phi}$ satisfies the recursion relations

$$\overset{0}{\Phi}X = \Lambda^t \overset{0}{\Phi} \tag{2.19}$$

we have the intertwining relation

$$A^{\phi}\Lambda^t = JA^{\phi}. \tag{2.20}$$

Since the infinite matrix A^{ϕ} is the sum of the identity matrix $\mathbf{I}_{\infty}$ and a strictly upper triangular one, its inverse exists (and has elements that are polynomials in those of A^{ϕ}).

We may therefore solve (2.20) for the recursion matrix J

$$J = A^{\phi}\Lambda^t \left(A^{\phi}\right)^{-1}. \tag{2.21}$$

In what follows, it will also be useful to define

$$\tilde{J} := A^{\phi}\Lambda \left(A^{\phi}\right)^{-1}, \tag{2.22}$$

which is a right inverse of J

$$J\tilde{J} = \mathbf{I}_{\infty}, \tag{2.23}$$

but not quite a left inverse

$$\tilde{J}J = \mathbf{I}_{\infty} - \mathbf{a}\mathbf{b}_1^t. \tag{2.24}$$

Here $\mathbf{b}_1^t$ is the seminfinite unit row vector $(\cdots, 0, 0, 1)$ and $\mathbf{a}$ is the right-most column vector of A^{ϕ}

$$\mathbf{a} = A^{\phi}\mathbf{b}_1. \tag{2.25}$$

For $k \in \mathbf{N}$, define the n-dimensional projection operator

$$\begin{aligned} \Pi_k : \mathbf{F} \otimes_{\mathbf{C}} \mathcal{H}_+ &\rightarrow \mathrm{span}\{\mathbf{b}_1, \dots \mathbf{b}_n\} \\ \Pi_k : \mathbf{b}_{j+k} &\mapsto \mathbf{b}_j \quad \text{if} \quad 1 \leq j \leq n \\ \Pi_k : \mathbf{b}_{j+k} &\mapsto \mathbf{0} \qquad \text{otherwise.} \end{aligned} \tag{2.26}$$

In the $\{\mathbf{b}_i\}$ basis, this is represented by the semi-infinite $n \times \infty$ matrix Γ_k whose column vectors all vanish, except in the successive (descending, according to our labelling) positions: $(n+k, \dots k+1)$, where they form the $n \times n$ identity matrix $\mathbf{I}_n$.

The projection of the $\infty \times n$ matrix W onto its kth $n \times n$ block is denoted

$$W(k) := \Gamma_k W = W_{(k^n)} \tag{2.27}$$

(where the RHS expresses the fact that this is just the $n \times n$ minor corresponding to the $n \times k$ rectangular partition (k^n)). Similarly, the restriction and projection of the $\infty \times \infty$ matrix J to the kth n-dimensional subspace is given by the $n \times n$ matrices

$$J(k) := \Gamma_k J \Gamma_k^t = \Lambda_n^t + J^+(k) \tag{2.28}$$

where

$$J^+(k) := \Gamma_k J^+ \Gamma_k^t \tag{2.29}$$

and Λ_n^t is the $n \times n$ lower triangular shift matrix.

Applying this to Φ, we obtain

$$\Phi(k) = \begin{pmatrix} \phi_{n+k-1}(x_1) & \phi_{n+k-1}(x_2) & \cdots & \phi_{n+k-1}(x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{k+1}(x_1) & \phi_{k+1}(x_2) & \cdots & \phi_{k+1}(x_n) \\ \phi_k(x_1) & \phi_k(x_2) & \cdots & \phi_k(x_n) \end{pmatrix}. \tag{2.30}$$

In particular

$$\Phi(0) = \begin{pmatrix} \phi_{n-1}(x_1) & \phi_{n-1}(x_2) & \cdots & \phi_{n-1}(x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_n) \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{2.31}$$

Note that, since all ϕ_k's are monic, the determinant of $\Phi(0)$ is just the Vandermonde determinant

$$\det(\Phi(0)) = \Delta(x_1, \dots x_n) = \prod_{i<j}(x_i - x_j). \tag{2.32}$$

Projecting onto the kth n-dimensional subspace, the recursion relations (2.14) give the following n-fold sequence of relations, valid for all $k \in \mathbf{N}$

$$\Phi(k)X = J^+(k)\Phi(k) + \Phi(k+1), \quad k = 0, 1, 2, \dots. \tag{2.33}$$

Now define, as in [25], the following generalizations of the usual Schur functions $S_\lambda(x_1, \dots, x_n)$,

$$S^\phi_\lambda(x_1, \dots, x_n) := \frac{\det(\Phi_\lambda)}{\det(\Phi(0))} = \frac{\pi_\lambda(\Phi)}{\pi_0(\Phi)}. \tag{2.34}$$

Remark 2.1 The classical case, for which (2.34) becomes the usual bi-alternant formula of Jacobi, corresponds to choosing $\phi_i(x) := x^i$, for which $J^+ = 0.$ and $J = \Lambda^t$. In [25], the case of orthogonal polynomials, for which the recursion matrix is J is tridiagonal, was considered. No such restriction is needed in the following; the results hold for arbitrary polynomial systems.

From the generalized Giambelli identity (2.9) and the fact that the S^ϕ_λ are Plücker cordinates of the element $[\Phi]$, with $S^\phi_0(x_1, x_2 \dots, x_n) = 1$, we have the Giambelli identity for generalized Schur functions

$$S^\phi_\lambda = \det\left(S^\phi_{(a_i|b_j)}\right)|_{1\leq i,j\leq \ell(\lambda)}, \tag{2.35}$$

where $(a_1 \dots a_k | b_1 \dots b_k)$ is λ in Frobenius notation.

2.2 The generalized Jacobi-Trudi formula: first form

The analogs of the *complete symmetric functions* are denoted

$$h^{(0)}_i := S^\phi_{(i)} \quad \text{for} \quad i \geq 0, \tag{2.36}$$

$$h^{(0)}_{-i} := 0 \quad 1 \leq i \leq n-1. \tag{2.37}$$

We may view these as the components of the column $H^{(1)}$ of an $\infty \times \infty$ matrix $\mathbf{H}$, whose elements are denoted

$$H_{ij} := h^{(j-1)}_{i-n}, \quad i, j \in \mathbf{Z}, \quad -\infty < j \leq n, \quad 1 \leq i \leq \infty. \tag{2.38}$$

The labelling conventions are such that the jth column is $H^{(j)}$, with j increasing from left to right consecutively from $-\infty$ to n. The components of the column vector $H^{(j)}$ are thus

$$H^{(j)}_i := H_{ij}, \quad 1 \leq i \leq \infty \tag{2.39}$$

with the first column given by

$$H^{(1)}_i = h^{(0)}_{i-n}. \tag{2.40}$$

The rows are labelled consecutvely, with i increasing upward from 1 to ∞, starting at the bottom. The successive columns $H^{(2)} \dots H^{(n)}$ are defined from the $H^{(1)}$ using the same recursion relations as those satisfied by the polynomial sequence $\{\phi_i\}_{i\in\mathbf{N}}$

$$H^{(j)} := JH^{(j-1)} = J^{j-1}H^{(1)}, \quad 1 \leq j \leq n. \tag{2.41}$$

Equivalently,

$$h^{(j+1)}_{i-n} = h^{(j)}_{i+1-n} + \sum_{k=1}^{i} J_{ik} h^{(j)}_{k-n} \tag{2.42}$$

The columns $H^{(j)}$ with $j \leq 0$ are also defined so the recursion relations (2.41) hold

$$H^{(j-1)} := \tilde{J} H^{(j)} \tag{2.43}$$

Multiplying on the left by J shows that the recursion relation (2.41) also holds for $j \leq 0$. It follows from (2.37) that

$$h^{(j)}_{-j} = 1, \quad h^{(j)}_{-k} = 0 \quad \text{if} \quad k > j. \tag{2.44}$$

Let H denote the $\infty \times n$ matrix consisting of the first n columns of $\mathbf{H}$:

$$H := \left(H^{(1)} \ H^{(2)} \dots H^{(n)} \right). \tag{2.45}$$

As before, denote by $H(k)$ the $n \times n$ minor obtained by projecting onto the n-dimensional subspace span$(\mathbf{b}_{n+k-1}, \dots \mathbf{b}_k)$

$$H(k) := \Gamma_k H = H_{(k^n)}, \quad k = 0, 1, \dots \tag{2.46}$$

and let $H^{(j)}(k)$ denote its jth column vector. The recursion relations (2.41) can equivalently be written

$$H^{(j+1)}(k) = J^+(k) H^{(j)}(k) + H^{(j)}(k+1). \tag{2.47}$$

The column vectors of $H(0)$ are

$$H^{(j)}(0) = \begin{pmatrix} h^{(j-1)}_0 \\ \vdots \\ h^{(j-1)}_{-j+1} = 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} := h^{(j-1)}. \tag{2.48}$$

Therefore

$$H(0) = \left(h^{(0)}\ h^{(1)} \dots h^{(n-1)}\right) \tag{2.49}$$

is upper triangular with 1's along the diagonal and hence has unit determinant

$$\det\left(H(0)\right) = 1. \tag{2.50}$$

Note that, because of (2.44), the Plücker coordinates $\pi_\lambda(H)$ of the element H may equivalently be written as $\ell(\lambda) \times \ell(\lambda)$ determinants:

$$\pi_\lambda(H) = \det(H_\lambda) = \det \begin{pmatrix} h^{(0)}_{\lambda_1} & h^{(1)}_{\lambda_1} & \cdots & h^{(\ell(\lambda)-1)}_{\lambda_1} \\ \vdots & \vdots & \cdots & \vdots \\ h^{(0)}_{\lambda_{\ell(\lambda)}-\ell(\lambda)+1} & h^{(1)}_{\lambda_{\ell(\lambda)}-\ell(\lambda)+1} & \cdots & h^{(\ell(\lambda)-1)}_{\lambda_{\ell(\lambda)}-\ell(\lambda)+1} \end{pmatrix}. \tag{2.51}$$

We are now ready to state the main result, which is a generalization of the one obtained for the case of orthogonal polynomials in [25]. (Cf. also [5] for the general setting of generalized Jacobi-Trudi identities associated to MKP τ-functions.)

Proposition 2.1 (Quantum Jacobi-Trudi identity) The semi-infinite matrices H and Φ represent the same element $[H] = [\Phi] \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ of the Grassmannian. Since both $\Phi(0)$ and $H(0)$ are invertible, this means

$$HH(0)^{-1} = \Phi\Phi(0)^{-1}. \tag{2.52}$$

It follows that their Plucker coordinates coincide:

$$\det\left(H_\lambda\right) = \frac{\det\left(\Phi_\lambda\right)}{\det\left(\Phi_0\right)} = S^\phi_\lambda, \tag{2.53}$$

which is equivalent to the generalized Jacobi-Trudi identity

$$S^\phi_\lambda = \det\left(h^{(j-1)}_{\lambda_i-i+1}\right)|_{1\leq i,j\leq \ell(\lambda)}. \tag{2.54}$$

Proof: Eq. (2.52) is equivalent to the set of equations

$$\Phi(k)\Phi(0)^{-1}H(0) = H(k), \quad k = 0, 1, \dots \tag{2.55}$$

or, by columns

$$\Phi(k)\Phi(0)^{-1}H^{(j)}(0) = H^{(j)}(k), \quad k = 0, 1, \dots, \quad 1 \leq j \leq n. \tag{2.56}$$

We prove this by (finite) induction on j, for $1 \leq j \leq n$. The $j = 1$ case

$$\Phi(k)\Phi(0)^{-1}H^{(1)}(0) = H^{(1)}(k), \quad \forall k = 0, 1, \ldots \tag{2.57}$$

is satisfied since, by Cramer's rule, it is equivalent to the definition (2.36) of $h_i^{(0)}$.

Now assume (2.56) holds for $j - 1$

$$\Phi(k)\Phi(0)^{-1}H^{(j-1)}(0) = H^{(j-1)}(k), \quad k = 0, 1, \ldots. \tag{2.58}$$

From the recursion relations (2.33) and (2.47) for $k = 0$, we have

$$H^{(j)}(0) = H^{(j-1)}(1) - \Phi(1)\Phi^{-1}(0)H^{(j-1)}(0) + \Phi(0)X\Phi^{-1}(0)H^{(j-1)}(0).t \tag{2.59}$$

By the inductive hypothesis, the first two terms cancel, so we have

$$H^{(j)}(0) = \Phi(0)X\Phi^{-1}(0)H^{(j-1)}(0). \tag{2.60}$$

Substituting this in the LHS of (2.56) gives

$$\begin{aligned}
\Phi(k)\Phi^{-1}(0)H^{(j)}(0) &= \Phi(k)X\Phi^{-1}(0)H^{(j-1)}(0) \\
&= \Phi(k+1)\Phi^{-1}(0)H^{(j-1)}(0) + J(k)\Phi(k)\Phi^{-1}(0)H^{(j-1)} \\
&= H^{(j-1)}(k+1) + J(k)H^{(j-1)}(k) \\
&= H^{(j)}(k),
\end{aligned} \tag{2.61}$$

where the recursion relation (2.33) was used in the second line, the inductive hypothesis in the third, and the recursion relation (2.47) in the fourth. QED

We now give an alternative proof of Proposition (2.1), which sheds further light on the meaning of the elements $[\Phi] = [H]$. For the case of monomials, denote the matrix H as $\overset{0}{H}$. We then have the following:

Lemma 2.1

$$H = A^{\phi}\overset{0}{H} \tag{2.62}$$

or equivalently, column by column

$$H^{(j)} = A^{\phi}\overset{0}{H}{}^{(j)}, \quad j = 1, \ldots n. \tag{2.63}$$

The proof is again inductive in the columns. For $j = 1$, by elementary row operations,

$$\det \begin{pmatrix} \phi_{n+k}(x_1) & \phi_{n+k}(x_2) & \cdots & \phi_{n+k}(x_n) \\ \phi_{n-2}(x_1) & \phi_{n-2}(x_1) & \cdots & \phi_{n-2}(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1(x_1) & \phi_1(x_1) & \cdots & \phi_1(x_n) \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$
=\det \begin{pmatrix} \phi_{n+k}(x_1) & \phi_{n+k}(x_2) & \cdots & \phi_{n+k}(x_n) \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}
$$

$$
= \sum_{i=0}^{n+k} A^{\phi}_{n+k+1,i+1} h_i \Delta(x_1, \ldots, x_n), \tag{2.64}
$$

where

$$
h_i := \frac{\det \begin{pmatrix} x_1^{n+i-1} & x_2^{n+i-1} & \cdots & x_n^{n+i-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\Delta(x_1, \ldots x_n)} \tag{2.65}
$$

are the complete symmetric functions, which are the components of $\overset{0}{H}{}^{(0)}$.

Now assume relation (2.63) holds for $j-1$,

$$
H^{(j-1)} = A^{\phi} \overset{0}{H}{}^{(j-1)}. \tag{2.66}
$$

Multiplying both sides on the left by J, and using (2.20) gives

$$
H^{(j)} = JH^{(j-1)} = JA^{\phi} \overset{0}{H}{}^{(j-1)} = A^{\phi} \Lambda^t \overset{0}{H}{}^{(j-1)} = A^{\phi} \overset{0}{H}{}^{(j)}. \tag{2.67}
$$

From this, it follows that

$$
H = \Phi T, \tag{2.68}
$$

where

$$
T = \Phi(0)^{-1} H(0) = \overset{0}{\Phi}(0)^{-1} \overset{0}{H}(0). \tag{2.69}
$$

This shows, in particular, that the change of basis matrix $\Phi(0)^{-1}H(0)$ is independent of the polynomial system $\{\phi_i(x)\}$ chosen. The invertible upper triangular matrix A^{ϕ} may be viewed as defining a group element $G_{\Phi} \in GL(\mathcal{H}_+, \mathbf{F})$ that carries the bases $\overset{0}{H}$ and $\overset{0}{\Phi}$ into H and Φ, respectively.

Remark 2.2 Eq. (2.62) could have been taken as the starting definition of H, from which the recursion relations (2.41) follow as a consequence of the intertwining relation (2.20).

In the context of integrable systems, the intertwining relation (2.20) may be viewed as a "dressing" transformation, that produces the initial value of the Lax matrix J from the "bare" (or vacuum) solution flowing from Λ^t. The dynamics are determined by an abelian group action on the Grassmannian, which induces commuting flows of KP type, as well as an isospectral flow of generalized Toda type for J [1]

2.3 The generalized Jacobi-Trudi formula: dual form

We now give a dual form of (2.54) that can be written in terms of the conjugate partition and analogs of the *elementary symmetric functions*. Let $\overset{0}{\tilde{E}}$ denote the $\infty \times \infty$ matrix with row vectors denoted $\overset{0}{E}_{(i)}$, $i \in \mathbf{Z}, -\infty < i \leq n$, with i decreasing from n to $-\infty$ vertically upward.

$$\overset{0}{E} := \begin{pmatrix} \vdots \\ \overset{0}{E}_{(n-1)} \\ \overset{0}{E}_{(n)} \end{pmatrix}. \tag{2.70}$$

The components of the row vectors $\overset{0}{E}_{(i)}$ are defined to be

$$\overset{0}{E}{}^j_{(i)} = (-1)^{n-i-j+1} e_{n-i-j+1}, \quad j \in \mathbb{N}, \quad -\infty < i \leq n, \tag{2.71}$$

where e_i is the ith *elementary symmetric function*. The ordering is such that the column index j decreases from left to right, ending with $j = 1$ (i.e., the transpose of the convention for matrices $\overset{0}{\mathbf{H}}$ and $\mathbf{H}$). Note that the bottom row $\overset{0}{E}_{(n)}$ is just the semi-infinite row vector

$$\overset{0}{E}_{(n)} = (\ldots, 0, 0, 1) = \mathbf{b}_1^t. \tag{2.72}$$

It follows from (2.71) that the rows satisfy the recursion relations

$$\overset{0}{E}_{(i)} = \overset{0}{E}_{(i+1)} \Lambda^t + \overset{0}{E}_{(n)} (-1)^{n-i} e_{n-i} \tag{2.73}$$

$$\overset{0}{E}_{(i+1)} = \overset{0}{E}_{(i)} \Lambda. \tag{2.74}$$

Now define the $\infty \times \infty$ matrix

$$\mathbf{E} := \overset{0}{\mathbf{E}} (A^\phi)^{-1} \tag{2.75}$$

whose rows are similarly denoted $E_{(i)}$, $i \in \mathbf{Z},\ i \leq n$, and whose elements are

$$E^j_{(i)} = E_{ij} := (-1)^{n-i-j+1} e^{n-j+1}_{(-i)}. \tag{2.76}$$

Note that, because of the upper triangular form of A^ϕ, we have

$$E_{(n)} = \overset{0}{E}_{(n)} = (\dots, 0, 0, 1). \tag{2.77}$$

Proposition 2.2 The rows of E satisfy the recursion relations

$$E_{(i)} = E_{(i+1)}J + E_{(n)}(-1)^{n-i}e_{n-i} \tag{2.78}$$
$$E_{(i+1)} = E_{(i)}\tilde{J}, \quad -\infty < i \leq n-1. \tag{2.79}$$

Proof: From (2.75), and (2.73) we have

$$\begin{aligned} E_{(i)} &= \overset{0}{E}_{(i)}(A^\phi)^{-1} = (\overset{0}{E}_{(i+1)}\Lambda^t + \overset{0}{E}_{(n)}(-1)^{n-i}e_{n-i})(A^\phi)^{-1} \\ &= E_{(i+1)}A^\phi\Lambda^t(A^\phi)^{-1} + E_{(n)}(-1)^{n-i}e_{n-i} \qquad (2.80) \\ &= E_{(i+1)}J + E_{(n)}(-1)^{n-i}e_{n-i}. \qquad (2.81) \end{aligned}$$

From (2.75), and (2.74) we have

$$E_{(i+1)} = \overset{0}{E}_{(i+1)}(A^\phi)^{-1} = \overset{0}{E}_{(i)}\Lambda(A^\phi)^{-1} = E_{(i)}A^\phi\Lambda^t(A^\phi)^{-1} = E_{(i)}\tilde{J}. \tag{2.82}$$

Corollary 2.1 *The matrices* $\mathbf{E}$ *and* $\mathbf{H}$ *are mutual inverses*

$$\mathbf{HE} = \mathbf{EH} = \mathbf{I}_\infty. \tag{2.83}$$

Proof: The complete and elementary symmetric functions satisfy the orthogonality relations

$$\sum_{k=i}^{j}(-1)^{i-k}e_{i-k}h_{k-j} = \delta_{ij}, \tag{2.84}$$

which imply that $\overset{0}{\mathbf{H}}$ and $\overset{0}{\mathbf{E}}$ are mutually inverse

$$\overset{0}{\mathbf{H}}\overset{0}{\mathbf{E}} = \overset{0}{\mathbf{E}}\overset{0}{\mathbf{H}} = \mathbf{I}_\infty. \tag{2.85}$$

Eq. (2.83) then follows from

$$\mathbf{H} = A^\phi\overset{0}{\mathbf{H}}, \quad \mathbf{E} = \overset{0}{\mathbf{E}}(A^\phi)^{-1}. \tag{2.86}$$

The subspace spanned by $\{\overset{0}{E}_{(0)}, \overset{0}{E}_{(-1)}, \cdots\}$ may be viewed as the element $W^* \in \mathrm{Gr}^*_n(\mathcal{H}_+, \mathbf{F})$ of the dual Grassmannian that annihilates the element $W \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$

spanned by $\{H^{(1)}, \cdots, H^{(n)}\}$. It follows [10] that the Plücker coordinates, $\pi_\lambda(W)$ and $\pi_{\lambda'}(W^*)$ are related by

$$\begin{aligned}
\det\left(h^{(j-1)}_{\lambda_i-i+1}\right)|_{1\le i,j\le \ell(\lambda)} &= \frac{\pi_\lambda(W)}{\pi_0(W)} = (-1)^{|\lambda|}\frac{\pi_{\lambda'}(W^*)}{\pi_0(W^*)} \\
&= (-1)^{|\lambda|}\det\Big((-1)^{\lambda'_j-j+i}e^{\lambda'_j-j+1}_{(i-1)}\Big)|_{1\le i,j\le l(\lambda')} \\
&= \det\Big(e^{\lambda'_j-j+1}_{(i-1)}\Big)|_{1\le i,j\le l(\lambda')}, \qquad (2.87)
\end{aligned}$$

which yields the dual form of the generalized Jacobi-Trudi identity,.

Proposition 2.3 (Dual generalized Jacobi-Trudi identity)

$$S^\phi_\lambda = \det\Big(e^{\lambda'_j-j+1}_{(i-1)}\Big)|_{1\le i,j\le l(\lambda')}. \qquad (2.88)$$

Corollary 2.2 *The elements of the row $E_{(0)}$ are equal to the generalized elementary symmetric functions $S^\phi_{(1)^{n-j+1}}$*

$$E^j_{(0)} = e^{n-j+1}_{(0)} = S^\phi_{(1)^{n-j+1}}, \quad 1\le j\le n. \qquad (2.89)$$

Proof: This follows from (2.88) by choosing the partition $\lambda = (1)^{n-j+1}$, and noting the dual partition $\lambda' = (j)$ has length 1.

3 Integrable systems and τ-functions

3.1 The polynomials S^ϕ_λ as KP τ-functions

We now show that the generalized Schur polynomials $S^\phi_\lambda([x])$, when viewed as functions of the monomial sums

$$t_i := \frac{1}{i}\sum_{a=1}^n x_a^i, \qquad (3.1)$$

are KP τ-functions, in the sense of Sato [22, 23] and Segal and Wilson [24]. We use the standard notation

$$[x] = \mathbf{t} = (t_1, t_2, \dots) \qquad (3.2)$$

to denote the infinite sequence of monomials sums defined in (3.1). First, we define coefficients $\phi^{(n)}_{\lambda\mu}$ as the Plücker coordinates $\pi_\mu(C^{(\lambda,n)})$ of the element $C^{(\lambda,n)}$ of the Grassmannian $\mathrm{Gr}_n(\mathcal{H}_+, \mathbf{C})$ spanned by the polynomials $\{\phi_{\lambda_i-i+n}(z)\}_{i=1,\dots n}$

$$\phi^{(n)}_{\lambda\mu} := \pi_\mu(C^{(\lambda,n)}) = \det\left(\phi_{l_i,m_j}\right)|_{1\le i,j\le n}, \qquad (3.3)$$

where

$$l_i := \lambda_i - i + n, \quad m_j := \mu_j - j + n, \quad i, j \in \mathbf{N}^+, \tag{3.4}$$

are the particle coordinates associated to partitions λ and μ.

So far, we have not explicitly indicated the dependence on the integer n in the expression for the symmetric polynomials $S^\phi_\lambda([x])$. But henceforth, n will be viewed as an integer variable, and we use the following notation to indicate this

$$S^\phi_{\lambda,n}([x]) := S^\phi_\lambda(x_1, \dots x_n). \tag{3.5}$$

Remark 3.1 Schur polynomials $S_\lambda([x])$, when viewed as functions of the monomial sum variables (3.1), (3.2), do not depend on the number n of x_i's, provided this is at least equal to the length $\ell(\lambda)$ of the partition. (Otherwise, they vanish). However, for general $S^\phi_{\lambda,n}([x])$, this is not the case, even though these too may be viewed as functions of the monomial sum variables (3.1), (3.2). As will be seen explicitly below, the $S^\phi_{\lambda,n}([x])$'s do, in general, depend on the number n. (The n independence in the case of $\Phi = \overset{0}{\Phi}$ is due to the fact that, in this case, we have the equality

$$h^j_i = h^{(0)}_{j+i} = h_{j+i}, \tag{3.6}$$

where the h_i's are the usual complete symmetric functions.)

The following shows that the $\phi^{(n)}_{\lambda\mu}$'s are the coefficients in the expression for $S^\phi_{\lambda,n}([x])$ as a linear combination of Schur functions $S_\lambda([x])$.

Lemma 3.1

$$S^\phi_{\lambda,n}([x]) = \sum_{\substack{\mu \\ \ell(\mu)\le n \\ |\mu|\le|\lambda|}} \phi^{(n)}_{\lambda\mu} S_\mu([x]). \tag{3.7}$$

Proof: Let $C^{(\lambda,n)}$ be the $\infty \times n$ matrix whose ith column consists of the coefficients $\phi_{\lambda_i - i+n+1,j}$ of the polynomial $\phi_{\lambda_i-i+n+1}(z)$ (i.e., its representation as a column vector relative to the basis $\{\mathbf{b}_j := z^{j-1}\}_{j\in\mathbf{N}}$). By the Cauchy-Binet identity, we have

$$\det\left((C^{(\lambda,n)})^t \overset{0}{\Phi}\right) = \sum_{\substack{\mu \\ \ell(\mu)\le n}} \det\left((C^{(\lambda,n)})_\mu\right) \det(\overset{0}{\Phi}_\mu) = \sum_{\substack{\mu \\ \ell(\mu)\le n \\ |\mu|\le|\lambda|}} \pi_\mu(C^{(\lambda,n)}) S_\mu([x]). \tag{3.8}$$

Corollary 3.1 $S^\phi_{\lambda,n}(\mathbf{t})$ *is a KP τ-function.*

This follows from the fact that any expansion of the form (3.7) is a KP τ-function provided the coefficients $\phi_{\lambda\mu}$ satisfy the Plücker relations (with respect to μ, for fixed λ).

Remark 3.2 Note that, unlike ordinary Schur polynomials $S_\lambda([x])$, the polynomials $S^\phi_\lambda([x])$ do not necessarily vanish when $[x]=[\mathbf{0}]$ for $\lambda \neq (0)$. By Lemma 3.1, we have

$$S^\phi_\lambda(\mathbf{0}) = \phi^{(n)}_{\lambda,(0)} = \det\left(\phi_{\lambda_i - i+n, n-j}\right)|_{1\le i,j\le n}. \tag{3.9}$$

Remark 3.3 Through a subquotienting procedure [6], it is possible to identify the elements of the infinite Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}_- + \mathcal{H}_+)$ of Sato [22, 23]and Segal-Wilson [24] corresponding to these polynomial τ-functions. Equivalently, they can be expressed as vacuum state expectation values of products of fermionic operators, as explained in the next section.

3.2 Fermionic representation of τ-functions

The coefficients $\phi^{(n)}_{\lambda\mu}$ can be expressed in terms of fermionic creation and annihilation operators $\{\psi_i, \psi^\dagger_i\}_{i\in\mathbf{Z}}$ on a Fermi Fock space, satisfying the usual anticommutation relations

$$[\psi_i, \psi^\dagger_j]_+ = \delta_{ij}, \quad [\psi_i, \psi_j]_+ = 0, \quad [\psi^\dagger_i, \psi^\dagger_j]_+ = 0, \quad i, j \in \mathbf{Z}, \tag{3.10}$$

with the vacuum state $|0\rangle$ satisfying

$$\psi_i|0\rangle = 0, \quad i < 0, \quad \psi^\dagger_i|0\rangle, \quad i \geq 0. \tag{3.11}$$

For $n > 0$, let

$$|n\rangle := \psi_{n-1}\cdots\psi_0|0\rangle, \quad |-n\rangle := \psi^\dagger_{-n}\cdots\psi^\dagger_{-1}|0\rangle \tag{3.12}$$

be the charge n (or $-n$) vacuum state, and denote the basis states in the charge n sector (for n positive or negative)

$$|\lambda; n\rangle := \psi_{\lambda_1 - 1+n}\cdots\psi_{\lambda_{\ell(\lambda)}-\ell(\lambda)+n}|n-\ell(\lambda)\rangle. \tag{3.13}$$

(In particular, $|(0); n\rangle$ is just denoted $|n\rangle$.) Recall that in general a lattice of mKP τ-functions can be expressed [8, 18] as

$$\tau_g(n, \mathbf{t}) = \langle n|\gamma_+(\mathbf{t})g|n\rangle, \quad n \in \mathbf{Z}, \tag{3.14}$$

where

$$\gamma_+(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i J_i}, \quad J_i := \sum_{j\in\mathbf{Z}} \psi_j\psi^\dagger_{j+i} \tag{3.15}$$

and g is any element of the infinite Clifford algebra generated by (3.10) that satisfies the bilinear relation

$$\left[\sum_{i\in\mathbf{Z}} \psi_i \otimes \psi_i^\dagger, g \otimes g\right] = 0. \tag{3.16}$$

In particular, the Schur functions can be expressed fermionically either as

$$S_\lambda(\mathbf{t}) = \langle n|\gamma_+(\mathbf{t})|\lambda; n\rangle, \tag{3.17}$$

or equivalently as

$$S_\lambda(\mathbf{t}) = \langle \lambda; n|\gamma_-(\mathbf{t})|n\rangle, \tag{3.18}$$

where

$$\gamma_-(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i J_{-i}}. \tag{3.19}$$

A 2D-Toda chain of τ-functions can be represented fermionically [8] as

$$\tau_g(n, \mathbf{t}, \mathbf{s}) = \langle n|\gamma_+(\mathbf{t}) g \gamma_-(\mathbf{s})|n\rangle, \quad n \in \mathbf{Z}. \tag{3.20}$$

where

$$\mathbf{s} := (s_1, s_2, \dots) \tag{3.21}$$

is viewed as an additional infinite set of commuting flow variables.

By Wick's theorem, we have (cf. ref. [6])

$$\phi^{(n)}_{\lambda\mu} = \langle \mu; n| \prod_{i=1}^n w_i^\lambda |0\rangle, \tag{3.22}$$

where

$$w_i^\lambda := \sum_{j=0}^{\lambda_i - i + n} \phi_{\lambda_i - i + n, j} \psi_j. \tag{3.23}$$

We may also express $\phi^{(n)}_{\lambda\mu}$ in a more symmetrical form as follows. The matrix ϕ may be viewed as representing an element of the group of invertible automorphisms of the Hilbert space $\mathcal{H} := L^2(S^1)$ of square integrable functions on the unit circle $\{z|\ |z|^2 = 1\}$, which is the direct sum of the two subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ spanned, respectively, by the positive powers $\{z^i\}_{i\in\mathbf{N}}$ and negative powers $\{z^{-i}\}_{i\in\mathbf{N}^+}$ of z . The element represented by ϕ in this monomial basis, leaves each subspace $\mathcal{H}_\pm$ invariant, acts as the identity transformation within the subspace $\mathcal{H}_+$ and has matrix representation ϕ in the negative monomial basis $\{z^{-i}\}$ for $\mathcal{H}_-$. We may express this infinite lower triangular matrix with

coefficients $(\phi_{ij})_{i,j\in\mathbf{N}}$ that are all equal to 1 on the diagonal as the exponential of a strictly lower triangular matrix α

$$\phi = e^{\alpha} \tag{3.24}$$

with matrix elements $(\alpha_{ij})_{i,j\in\mathbf{N}}$ satisfying

$$\alpha_{ij} = 0 \quad \text{if} \quad j \geq i. \tag{3.25}$$

The fermionic representation [8, 12, 22] of this group element is then

$$g_{\phi} := \exp \sum_{i>j\geq 0} \alpha_{ij}\psi_i\psi_j^{\dagger}. \tag{3.26}$$

while the fermionic representation of the transpose

$$\phi^t = e^{\alpha^t} \tag{3.27}$$

is

$$g_{\phi^t} = \exp \sum_{i>j\geq 0} \alpha_{ij}\psi_j\psi_i^{\dagger}. \tag{3.28}$$

Note that g_{ϕ} stabilizes the left charge n vacuum and g_{ϕ^t} the right charge n vacuum for any n

$$\langle n|g_{\phi} = \langle n|, \quad g_{\phi^t}|n\rangle = |n\rangle \tag{3.29}$$

since, for $i > j \geq 0$

$$\psi_j\psi_i^{\dagger}|n\rangle = 0. \tag{3.30}$$

Lemma 3.2 The matrix elements A^{ϕ}_{ij} and J_{ij} may be expressed fermionically as

$$A^{\phi}_{ij} = \phi_{i-1,j-1} = \langle (i);0|g_{\phi}|(j);0\rangle = \langle (i-k);k|g_{\phi}|(j-k);k\rangle, \tag{3.31}$$

$$J_{ij} = \langle (i);0|g_{\phi}J_{-1}g_{\phi}^{-1}|(j);0\rangle = \langle (i-k);k|g_{\phi}J_{-1}g_{\phi}^{-1}|(j-k);k\rangle, \quad \forall\ k \leq i,j, \tag{3.32}$$

for any charge sector $k \leq i, j$.

Proof: Eq. (3.31) follows from eqs. (3.13), (3.29) and (3.26), which imply

$$\begin{aligned}\langle (i-k);k|g_{\phi}|(j-k);k\rangle &= \langle (0);k-1|\psi_{i-1}^{\dagger}g_{\phi}\psi_{j-1}|(0);k-1\rangle \\ &= \langle (0);k-1|g_{\phi}^{-1}\psi_{i-1}^{\dagger}g_{\phi}\psi_{j-1}|(0);k-1\rangle \\ &= \sum_{l=0}^{i-1}\phi_{i-1,l}\langle (0);k-1|\psi_l^{\dagger}\psi_{j-1}|(0);k-1\rangle.\end{aligned}$$

$$= \phi_{i-1,j-1} = A^{\phi}_{ij}. \tag{3.33}$$

Eq. (3.32) follows by substituting

$$g_\phi J_{-1} g_\phi^{-1} = \sum_{l\in\mathbf{Z}} g_\phi \psi_l g_\phi^{-1} g_\phi \psi^\dagger_{l-1} g_\phi^{-1} = \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l-1} \phi_{kl}\phi^{-1}_{l-1,m}\psi_k\psi^\dagger_m \tag{3.34}$$

into

$$\begin{aligned}\langle (i-k);k|g_\phi J_{-1} g_\phi^{-1}|(j-k);k\rangle &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l-1} \phi_{kl}\phi^{-1}_{l-1,m}\langle (i-k);k|\psi_k\psi^\dagger_m|(j-k);k\rangle \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l-1} \phi_{kl}\phi^{-1}_{l-1,m}\langle (0);k-1|\psi^\dagger_{i-1}\psi_k\psi^\dagger_m\psi_{j-1}|(0);k-1\rangle \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{l-1} \phi_{i-1,l}\phi^{-1}_{l-1,j-1} = \left(A^\phi\Lambda^t(A^\phi)^{-1}\right)_{ij} = J_{ij},\end{aligned} \tag{3.35}$$

where the third equality follows from Wick's theorem.

More generally, we have the following expression for $\phi^{(n)}_{\lambda\mu}$ as a fermionic matrix element

Proposition 3.1

$$\phi^{(n)}_{\lambda\mu} = \langle\lambda;n|g_\phi|\mu;n\rangle = \langle\mu;n|g_{\phi^t}|\lambda;n\rangle. \tag{3.36}$$

Proof:[†]

It follows from the canonical anticommutation relations (3.10), that

$$g_{\phi^t}\psi_i g_{\phi^t}^{-1} = \sum_{j=0}^{i}\phi_{ij}\psi_j. \tag{3.37}$$

Therefore, from eq. (3.23), we have

$$w_i^\lambda := \sum_{j=0}^{\lambda_i-i+n}\phi_{\lambda_i-i+n,j}\psi_j = g_{\phi^t}\psi_{\lambda_i-i+n}g_{\phi^t}^{-1}. \tag{3.38}$$

The expression (3.13) for $|\lambda;n\rangle$ can equivalently be written

$$|\lambda;n\rangle := \psi_{\lambda_1-1+n}\cdots\psi_{\lambda_n}|0\rangle. \tag{3.39}$$

[†]This proof was suggested by A. Yu. Orlov. See also [13], eq. (1.34) for a related fermionic identity.

Therefore, from (3.22),

$$\phi^{(n)}_{\lambda\mu} = \langle \mu; n|w^{\lambda}_1 \dots w^{\lambda}_n|n\rangle = \langle \mu; n|g_{\phi^t}|\lambda; n\rangle = \langle \lambda; n|g_\phi|\mu; n\rangle. \tag{3.40}$$

QED

From (3.18) and (3.36), we obtain a fermionic expression for $S^{\phi}_{\lambda,n}([x])$.

Corollary 3.2

$$S^{\phi}_{\lambda,n}([x]) = \langle \lambda; n|g_\phi \gamma_-([x])|n\rangle = \langle n|\gamma_+([x])g_{\phi^t}|\lambda; n\rangle. \tag{3.41}$$

Moreover, the polynomials $S^{\phi}_{\lambda}([x])$ can themselves be used as coefficients in a Schur function expansion to define a family of KP τ-functions, in which the indeterminates $(x_1, \dots x_n)$ are interpreted as complex parameters

$$\tau_\phi(n, \mathbf{t}, [x]) := \sum_\lambda S^{\phi}_{\lambda,n}([x])S_\lambda(\mathbf{t}). \tag{3.42}$$

Here, the KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$ are independent and

$$\mathbf{s} := (s_1, s_2, \dots) := [x] \tag{3.43}$$

may be viewed as a second set of flow parameters. Then $\tau_\phi(n, \mathbf{t}, \mathbf{s})$ is simultaneously a KP τ-function in both $\mathbf{s}$ and $\mathbf{t}$ variables and, viewing n as a lattice variable, a 2D Toda τ-function.

Proposition 3.2 The functions $\tau_\phi(n, \mathbf{t}, \mathbf{s})$ form a 2D-Toda chain of τ-functions which may be expressed fermionically as

$$\tau_\phi(n, \mathbf{t}, \mathbf{s}) = \langle n|\gamma_+(\mathbf{t})g_\phi\gamma_-(\mathbf{s})|n\rangle. \tag{3.44}$$

Proof: We substitute (3.7) in (3.42) to express $\tau_\phi(n, \mathbf{t}, \mathbf{s})$ as a double Schur function expansion

$$\tau_\phi(n, \mathbf{t}, \mathbf{s}) = \sum_\lambda \sum_\mu \phi^{(n)}_{\lambda\mu} S_\lambda(\mathbf{t})S_\mu(\mathbf{s}). \tag{3.45}$$

Using eq. (3.36) and (3.18) gives

$$\begin{aligned}\tau_\phi(n, \mathbf{t}, \mathbf{s}) &= \sum_\lambda \sum_\mu \langle n|\gamma_+(\mathbf{t})|\lambda; n\rangle\langle \lambda; n|g_\phi|\mu; n\rangle\langle \mu; n|\gamma_-(\mathbf{s})|n\rangle \\ &= \langle n|\gamma_+(\mathbf{t})g_\phi\gamma_-(\mathbf{s})|n\rangle,\end{aligned} \tag{3.46}$$

which is the standard fermionic form [8, 13] for a chain of 2D-Toda τ-functions. QED

If all the parameters $\mathbf{s}$ in (3.44) are set equal to zero, we obtain the chain of KP τ-functions

$$\tau_\phi(n, \mathbf{t}, \mathbf{0}) = \langle n|\gamma_+(\mathbf{t}) g_\phi |n\rangle = \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} \phi^{(n)}_{\lambda,(0)} S_\lambda(\mathbf{t}). \tag{3.47}$$

Finally, we may choose a pair of polynomial systems $\{\phi_i\}_{i\in\mathbf{N}}$ and $\{\theta_i\}_{i\in\mathbf{N}}$ and associate to them the generalized Schur functions $S^\phi_{\lambda,n}$ and $S^\theta_{\lambda,n}$. Forming the sum of their products

$$\tau_{\phi,\theta}(n, \mathbf{t}, \mathbf{s}) := \sum_\lambda S^\phi_{\lambda,n}(\mathbf{s}) S^\theta_{\lambda,n}(\mathbf{t}), \tag{3.48}$$

we obtain a more general form of 2D Toda τ-functions.

Proposition 3.3

$$\tau_{\phi,\theta}(n, \mathbf{t}, \mathbf{s}) = \langle n|\gamma_+(\mathbf{t}) g^t_\theta g_\phi \gamma_-(\mathbf{s})|n\rangle. \tag{3.49}$$

Proof: Substitute eq.(3.41) in its first form for $S^\phi_{\lambda,n}(\mathbf{s})$ and in its second form for $S^\theta_{\lambda,n}(\mathbf{t})$ and use

$$\sum_\lambda |\lambda; n\rangle\langle\lambda; n| = \Pi_{(n)} \tag{3.50}$$

where $\Pi_{(n)}$ is the orthogonal projection map to the charge n sector.

Remark 3.4 The product $g^t_\theta g_\phi$ appearing in (3.49) represents a lower/upper triangular factorization of an operator in which the diagonal elements in each term have coefficients 1. This restriction, which follows from choosing the polynomials systems ϕ and θ to be monic, can be dropped with no essential change in the resulting τ-functions, except for their normalization, since these represent the same elements of the Grassmannian. This amounts to mutiplying one of the factors g^t_θ or g_ϕ, either on the left or on the right, by a diagonal term of the form

$$g_0(\rho) := e^{\sum_{i=0}^\infty T_i \psi_i \psi_i^\dagger}, \tag{3.51}$$

where the coefficients $\{T_i\}_{i\in\mathbf{N}}$ determine the leading term coefficients of the polynomials

$$\phi_i(x) = e^{T_i} x^i + \mathcal{O}(x^{i-1}). \tag{3.52}$$

This represents a convolution product on $\mathcal{H} = L^2(S^1)$ with the (distributional) element ρ whose negative Fourier series part is $\rho_-(z) := \sum_{i=0}^\infty \rho_i z^{-i-1}$, and whose positive part is the distribution represented by $\rho_+(z) = \sum_{i=0}^\infty z^i$. Such convolution actions have been studied as symmetries of KP and 2D Toda τ-functions in [14]. The factor $g_0(\rho)$ may be placed on the left $g_0(\rho)g^t_\theta g_\phi$ or on the right $g^t_\theta g_\phi g_0(\rho)$ in (3.49) without altering the symmetric polynomials $S^\theta_{\lambda,n}$ or $S^\phi_{\lambda,n}$ except by normalization factors. If the diagonal factor is placed in the middle $g^t_\theta g_0(\rho) g_\phi$, this amounts to replacing one of the two series of polynomials ϕ or θ by their convolution product with ρ.

3.3 Fermionic representation of the matrices H and E

We now give fermionic representations of the matrices H and E appearing in the generalized Jacobi-Trudi formulae (2.54), (2.87).

Proposition 3.4 The matrix elements $H_i^{(j)}$ and $E_{(i)}^j$ are given as fermionic matrix elements as follows

$$H_i^{(j)} = \langle (i+j-n-1); n-j+1 | g_\phi \gamma_-([x]) | n-j+1 \rangle = h_{i-n}^{(j-1)} \tag{3.53}$$
$$E_{(i)}^j = (-1)^{n-i-j+1} \langle (1)^{n-i-j+1}; n-i | g_\phi \gamma_-([x]) | n-i \rangle = (-1)^{n-i-j+1} e_{(-i)}^{n-j+1}. \tag{3.54}$$

Proof: We first verify these expressions for the case of monomials; i.e. when $\Phi = \overset{0}{\Phi}$, and then show that eq. (2.63) of Lemma 2.1 is satisfied.

Setting $g_\phi = \mathbf{I}$ in (3.54) we get

$$\langle (i+j-n-1); n-j+1 | \gamma_-([x]) | n-j+1 \rangle = h_{i+j-n-1}([x]) = \overset{0}{H}{}_i^{(j)}. \tag{3.55}$$

From eq. (3.13) it follows that

$$\begin{aligned} g_{\phi^t} | (i+j-n-1); n-j+1 \rangle &= g_{\phi^t} \psi_{i-1} | n-j \rangle \\ &= \sum_{k=1}^{i} \phi_{i-1,k-1} | (k+j-n-1); n-j+1 \rangle, \end{aligned} \tag{3.56}$$

where (3.37) has been used in the second line. Therefore

$$\begin{aligned} \langle (i+j-n-1); n-j+1 | g_\phi \gamma_-([x]) | n-j+1 \rangle &= \sum_{k=1}^{i} A_{ik}^\phi \langle (k+j-n-1); n-j+1 | \gamma_-([x]) | n-j+1 \rangle \\ &= \sum_{k=1}^{i} A_{ik}^\phi \overset{0}{H}{}_k^{(j)} = H_k^{(j)}. \end{aligned} \tag{3.57}$$

To prove eq. (3.54) we proceed similarly. Setting $g_\phi = \mathbf{I}$ in (3.54) we get

$$(-1)^{n-i-j+1} \langle (1)^{n-i-j+1}; n-i | \gamma_-([x]) | n-i \rangle = (-1)^{n-i-j+1} e_{n-i-j+1} = \overset{0}{E}{}_{(i)}^{j}. \tag{3.58}$$

From eq. (3.13) it follows that

$$g_{\phi^t} | (1)^{n-i-j+1}; n-i \rangle = -g_{\phi^t} \psi_{j-1}^\dagger | (n-i+1) \rangle$$

$$= -\sum_{k=1}^{i} \phi^{-1}_{k-1,i-1} |(n-k-j+1); n-j+1\rangle, \tag{3.59}$$

where

$$g_{\phi^t} \psi^\dagger_{j-1} g^{-1}_{\phi^t} = \sum_{k=0}^{\infty} \psi^\dagger_{k-1} \phi^{-1}_{k-1,j-1} \tag{3.60}$$

has been used in the second line. Substituting (3.59) in (3.54) gives

$$\begin{aligned} &(-1)^{n-i-j+1} \langle (1)^{n-i-j+1}; n-i | g_\phi \gamma_-([x]) | n-i \rangle \\ &= \sum_{k=0}^{\infty} \phi^{-1}_{k-1,j-1} (-1)^{n-i-k+1} \langle (1)^{n-i-k+1}; n-i | \gamma_-([x]) | n-i \rangle \\ &= \sum_{k=0}^{\infty} \overset{0}{E}{}^k_{(i)} A^{-1}_{kj} = E^j_{(i)}. \end{aligned} \tag{3.61}$$

QED

Remark 3.5 Proposition 3.4 shows that the generalized Jacobi-Trudi formulae (2.54), (2.87) are special cases of those studied in [5], with the group element denoted G in formula (3.16) of [5] given by

$$G = g_\phi \gamma_-([x]). \tag{3.62}$$

4 Examples and applications

We close with some examples and applications of the above results.

4.1 Character expansions of classical groups.

As pointed out in [25], irreducible characters of the classical groups $O(2n)$, $O(2n+1)$ and $Sp(2n)$ can be viewed as examples of the generalized Schur functions studied here, corresponding to certain systems of orthogonal polynomials. In each case, we may view the parameters $(x_1, \dots, x_n)$ as coordinates on the maximal torus of a compact real form of the relevant group. The subgroup reductions

$$Sp(n) \supset U(n), \quad O(2n) \supset U(n), \quad O(2n+1) \supset U(n) \tag{4.1}$$

correspond to identifying the maximal torus of $Sp(n), O(2n)$ and $O(2n+1)$ with subgroups of the maximal torus of $U(2n)$ or $U(2n+1)$ consisting of elements of the form $\mathrm{diag}(x_1, \dots, x_n, x_1^{-1}, \dots x_n^{-1})$ or $\mathrm{diag}(x_1, \dots, x_n, x_1^{-1}, \dots x_n^{-1}, 1)$.

The systems of orthogonal polynomials $\{\phi_i^{Sp(2n)}(z)\}$, $\{\phi_i^{SO(2n)}(z)\}$ and $\{\phi_i^{SO(2n+1)}(z)\}$ are expressed in terms of the variables

$$z := x + x^{-1} \tag{4.2}$$

as follows:

$$\begin{aligned}
\phi_i^{Sp(2n)}(z) &= \sum_{j=0}^{i} x^{i-2j} = \sum_{j=0}^{i} \phi_{ij}^{Sp(2n)} z^j, \\
\phi_i^{SO(2n)}(z) &= x^i + x^{-i} = \sum_{j=0}^{i} \phi_{ij}^{SO(2n)} z^j, \\
\phi_i^{SO(2n+1)}(z) &= \sum_{j=0}^{2i} x^{i-j} = \sum_{j=0}^{i} \phi_{ij}^{SO(2n+1)} z^j,
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
&\phi_{2i,2j}^{Sp(2n)} = (-1)^{i+j} \binom{i+j}{i-j}, \quad \phi_{2i+1,2j+1}^{Sp(2n)} = (-1)^{i+j} \binom{i+j+1}{i-j}, \\
&\phi_{2i,2j+1}^{Sp(2n)} = \phi_{2i+1,2j}^{Sp(2n)} = 0, \\
&\phi_{2i,2j}^{SO(2n)} = (-1)^{i+j} \frac{i}{j} \binom{i+j-1}{i-j} \text{ if } j > 0, \quad \phi_{2i+1,2j+1}^{SO(2n)} = (-1)^{i+j} \frac{2i+1}{2j+1} \binom{i+j}{i-j}, \\
&\phi_{0,0}^{SO(2n)} = 1, \quad \phi_{2i,0}^{SO(2n)} = 2(-1)^i \text{ if } i > 0, \quad \phi_{2i,2j-1}^{SO(2n)} = \phi_{2i-1,2j}^{SO(2n)} = 0, \\
&\phi_{2i,2j}^{SO(2n+1)} = (-1)^{i+j} \binom{i+j}{i-j}, \quad \phi_{2i+1,2j+1}^{SO(2n+1)} = (-1)^{i+j} \binom{i+j+1}{i-j}, \\
&\phi_{2i,2j+1}^{SO(2n+1)} = (-1)^{i+j+1} \binom{i+j}{i-j-1}, \quad \phi_{2i+1,2j}^{SO(2n+1)} = (-1)^{i+j} \binom{i+j}{i-j}.
\end{aligned} \tag{4.4}$$

The recursion matrices $J^{Sp(2n)}$, $J^{SO(2n)}$ and $J^{SO(2n+1)}$ for these orthogonal polynomials are given by

$$\begin{aligned}
J_{ij}^{Sp(2n)} &= \delta_{i+1,j} + \delta_{i,j+1}, \\
J_{ij}^{SO(2n)} &= \delta_{i+1,j} + \delta_{i,j+1} + \delta_{i2}\delta_{j1}, \\
J_{ij}^{SO(2n+1)} &= \delta_{i+1,j} + \delta_{i,j+1} - \delta_{i1}\delta_{j1}.
\end{aligned} \tag{4.5}$$

As noted in [25], the corresponding generalized Schur functions coincide with the irreducible characters [19, 9]:

$$S_\lambda^{Sp(2n)}(z_1,\dots,z_n) = \chi_\lambda^{Sp(2n)}(x_1,\dots,x_n) \tag{4.6}$$
$$S_\lambda^{SO(2n)}(z_1,\dots,z_n) = \chi_\lambda^{SO(2n)}(x_1,\dots,x_n) \tag{4.7}$$
$$S_\lambda^{SO(2n+1)}(z_1,\dots,z_n) = \chi_\lambda^{SO(2n+1)}(x_1,\dots,x_n) \tag{4.8}$$

where

$$z_i := x_i + x_i^{-1}. \quad i = 1,\dots,n \tag{4.9}$$

and

$$S_\lambda^G(z_1,\dots,z_n) := S_\lambda^{\phi^G}(z_1,\dots,z_n) \tag{4.10}$$

for $G = Sp(2n),\ SO(2n),$ or $SO(2n+1)$. The generalised Jacobi-Trudi formula for these cases are equivalent, by elementary row operations, to the determintal formulae for these characters given in Props. 24.22, 24.33, 24.44 of [9].

For any pair of partitions (λ,μ), and $G = Sp(2n),\ SO(2n),$ or $SO(2n+1)$, define

$$\phi_{\lambda\mu}^G := \det\left(\phi^G_{\lambda_i - i + n, \mu_j - j + n}\right)_{1\le i,j,\le n}. \tag{4.11}$$

By Lemma (3.1) we have the expansions

$$S_\lambda^{Sp(2n)}(z_1,\dots,z_n) = \sum_{\substack{\ell(\mu)\le n\\ \mu\le|\lambda|}} \phi_{\lambda\mu}^{Sp(2n)} S_\mu(z_1,\dots,z_n), \tag{4.12}$$
$$S_\lambda^{SO(2n)}(z_1,\dots,z_n) = \sum_{\substack{\ell(\mu)\le n\\ \mu\le|\lambda|}} \phi_{\lambda\mu}^{SO(2n)} S_\mu(z_1,\dots,z_n), \tag{4.13}$$
$${}_\lambda^{SO(2n+1)}(z_1,\dots,z_n) = \sum_{\substack{\ell(\mu)\le n\\ \mu\le|\lambda|}} \phi_{\lambda\mu}^{SO(2n+1)} S_\mu(z_1,\dots,z_n). \tag{4.14}$$

This should be compared with Littlewood's character expansion formulae [19, 16]

$$\chi_\lambda^{Sp(2n)}(x_1,\dots,x_n) = \sum_{\substack{\ell(\mu)\le n\\ \mu\le|\lambda|}} \sum_\alpha (-1)^{|\alpha|} C^\lambda_{D'(\alpha),\mu} S_\mu(x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1}), \tag{4.15}$$
$$\chi_\lambda^{SO(2n)}(x_1,\dots,x_n) = \sum_{\substack{\ell(\mu)\le n\\ \mu\le|\lambda|}} \sum_\alpha (-1)^{|\alpha|} C^\lambda_{D(\alpha),\mu} S_\mu(x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1}), \tag{4.16}$$
$$\chi_\lambda^{SO(2n+1)}(x_1,\dots,x_n) = \sum_{\substack{\ell(\mu)\le n\\ \mu\le|\lambda|}} \sum_\alpha (-1)^{|\alpha|} C^\lambda_{D(\alpha),\mu} S_\mu(x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1},1) \tag{4.17}$$

where the sums in α are taken over strict partitions

$$\alpha = (\alpha_1, \dots \alpha_r), \quad \alpha_i \in \mathbf{N}^+, \quad \alpha_1 > \cdots > \alpha_r > 0, \quad 2|\alpha| \leq |\lambda|, \tag{4.18}$$

$D(\alpha)$ and $D'(\alpha)$ are their "doubles", defined in Frobenius notation by

$$\begin{aligned} D(\alpha) &:= (\alpha_1, \dots, \alpha_r | \alpha_1 - 1, \dots, \alpha_r - 1) \\ D'(\alpha) &= (\alpha_1 - 1, \dots, \alpha_r - 1 | \alpha_1, \dots, \alpha_r), \end{aligned} \tag{4.19}$$

and $C^{\lambda}_{\mu\nu}$ are the Littlewood-Richardson coefficients

4.2 Moment matrices and matrix models.

Suppose the infinite triangular matrix ϕ_{ij} defining our system of polynomials is chosen to be the one that provides a symmetric lower/upper symmetrical factorization of the Hankel matrix of moments

$$M_{ij} := \int_{\Gamma} d\mu(z) z^{i+j} \tag{4.20}$$

of some measure $d\mu$, supported on a curve Γ in the complex plane:

$$\sum_{k=\max(i,j)}^{\infty} \phi_{ki}\phi_{kj} = M_{ij}, \quad i, j \in \mathbf{N}. \tag{4.21}$$

It is well known [13] that the chain of KP τ-functions $\tau_{\phi,\phi}(n, \mathbf{t}, \mathbf{0})$ in (3.47) is then equal to the sequence of multiple integrals

$$\tau_{\phi,\phi}(n, \mathbf{t}, \mathbf{0}) = \frac{1}{n!} \left(\prod_{a=1}^{n} \int_{\Gamma} d\mu(z_a) \right) \Delta^2(\mathbf{z}) \exp\left(\sum_{j=1}^{\infty} \sum_{a=1}^{n} t_j z_a^j \right) \tag{4.22}$$

$$= \sum_{\lambda,\, \ell(\lambda) \leq n} B_{\lambda,n}(d\mu) S_{\lambda}(\mathbf{t}) \tag{4.23}$$

$$= \sum_{\lambda,\, \ell(\lambda) \leq n} S^{\phi}_{\lambda}(\mathbf{0}) S^{\phi}_{\lambda}(\mathbf{t}), \tag{4.24}$$

where

$$\Delta(\mathbf{z}) = \prod_{i<j}^{n} (z_i - z_j) \tag{4.25}$$

is the Vandermonde determinant and

$$B_{\lambda,n}(d\mu) := \det(M_{\lambda_i - i + j})_{1 \leq i,j \leq n} = \sum_{\nu,\, \ell(\nu) \leq n} \phi^{(n)}_{\nu\lambda} \phi^{(n)}_{\nu(0)}. \tag{4.26}$$

Within a normalization, (4.24) is just the reduced form of the partition function of a generalized conjugation invariant normal matrix model with spectrum supported on Γ, expressed as a multiple integral over the eigenvalues [3].

The matrix of moments can also be expressed in an upper/lower triangular symmetrically factorized form

$$M = \tilde{\phi}^{-1}(\tilde{\phi}^t)^{-1}, \tag{4.27}$$

where the polynomials

$$\tilde{\phi}_i(x) := \sum_{j=0}^{i} \tilde{\phi}_{ij} x^j, \quad i \in \mathbf{N} \tag{4.28}$$

formed from the components $\tilde{\phi}_{ij}$ of the matrix $\tilde{\phi}$ are orthogonal with respect to the measure $d\mu$

$$\int_\Gamma \tilde{\phi}_i(x)\tilde{\phi}_j(x) d\mu(x) = \delta_{ij}. \tag{4.29}$$

The equality

$$\phi^t \phi = \tilde{\phi}^{-1}(\tilde{\phi}^t)^{-1} \tag{4.30}$$

may be viewed as defining a Darboux transformation of the measure to the new measure $d\tilde{\mu}$ whose matrix of moments $\tilde{M}$ is

$$\tilde{M}_{ij} = \int_\Gamma d\tilde{\mu}(x) x^{i+j} = \sum_{k=\max(i,j)}^{\infty} (\tilde{\phi})^{-1}_{ki} (\tilde{\phi})^{-1}_{kj}. \tag{4.31}$$

4.3 Bimoments of two variable measures and 2-matrix models.

We may similarly choose the product of matrices $\theta^t \phi$ entering in the 2D-Toda chain of τ-functions (3.49) to be the upper/lower factorization of the matrix of bimoments

$$M_{ij} := \sum_{p,q} c_{pq} \int_{\Gamma_p} \int_{\tilde{\Gamma}_q} d\mu(z,w) z^i w^j = (\theta^t \phi)_{ij} \tag{4.32}$$

of a two-variable measure defined along a linear combination of products of curves $\Gamma_p \times \tilde{\Gamma}_q$ in the z and w planes.

The resulting 2D-Toda τ-function appearing in eq. (3.49) is then the partition function of a coupled two-matrix model reduced by a generalized Itzykson-Zuber-Harish-Chandra

identity to a $2n$-fold integral over the eigenvalues [12],

$$\tau_{\theta,\phi}(n,\mathbf{t},\mathbf{s}) = \frac{1}{n!}\prod_{a=1}^{n}\left(\sum_{p,q} c_{pq}\int_{\Gamma_a}\int_{\tilde{\Gamma}_b} d\mu(z_a,w_a)e^{\sum_{i=1}^{\infty}t_i z_a^I}e^{\sum_{j=1}^{\infty}s_j w_a^j}\right)\Delta(\mathbf{z})\Delta(\mathbf{w}) \tag{4.33}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}}\sum_{\substack{\nu \\ \ell(\nu)\leq n}} B_{\lambda,\nu,n}(d\nu)S_\lambda(\mathbf{t})S_\nu(\mathbf{s}) \tag{4.34}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} S_\lambda^{\psi}(\mathbf{t})S_\lambda^{\phi}(\mathbf{s}), \tag{4.35}$$

where

$$B_{\lambda,\nu,n}(d\mu) := \det(M_{\lambda_i-i+n,\,\nu_j-j+n})_{1\leq i.j\leq n} == \sum_{\rho,\,\ell(\rho)\leq n}\psi_{\rho\lambda}^{(n)}\phi_{\rho\nu}^{(n)}. \tag{4.36}$$

Assuming that all consecutive diagonal $n\times n$ minors of the matrix of bimoments are nonsingular, we have a system of associated biorthogonal polynomials [4]

$$\tilde{\phi}_i(x) = \sum_{j=0}^{i}\tilde{\phi}_{ij}x^j, \quad \tilde{\theta}_j(y) = \sum_{k=0}^{j}\tilde{\theta}_{jk}y^k, \tag{4.37}$$

which may be normalized to have equal leading coefficients

$$\tilde{\phi}_{jj} = \tilde{\theta}_{jj}, \tag{4.38}$$

satisfying

$$\sum_{p,q} c_{pq}\int_{\Gamma_p}\int_{\tilde{\Gamma}_q} d\mu\tilde{\phi}_i(x)\tilde{\theta}_j(y) = \delta_{ij}. \tag{4.39}$$

It follows that the matrix of moments may be expressed in factorized lower/upper triangular form as

$$M = \tilde{\phi}^{-1}(\tilde{\theta}^t)^{-1} \tag{4.40}$$

where $\tilde{\phi}$ and $\tilde{\theta}$ are the lower triangular matrices with components $\tilde{\phi}_{ij}$ and $\tilde{\theta}_{ij}$, respectively. We therefore have the equality

$$\tilde{\phi}^{-1}(\tilde{\theta}^t)^{-1} = \theta^t\phi, \tag{4.41}$$

which means that the matrix pair (θ^t,ϕ) are the Darboux transformation of the pair $(\tilde{\phi}^{-1},\tilde{\theta}^{t-1})$, leading to a new measure $d\tilde{\mu}(z,w)$, with bimoment matrix $\tilde{M}$

$$\tilde{M}_{ij} = \sum_{pq} c_{ab}\int_{\Gamma_p}\int_{\tilde{\Gamma}_q} d\tilde{\mu}(z,w)z^i w^j = \sum_{k=\max(i,j)}^{\infty}(\tilde{\theta}^{-1})_{ki}(\tilde{\phi}^{-1})_{kj}, \tag{4.42}$$

which is thus the Darboux transformation of the measure $d\mu(z,w)$.

4.4 Generating function for nonintersecting random walks

The unnormalized transition rate for a continuous time, n-particle, right-moving, nearest neighbour exclusion process on the positive integer lattice can be expressed as the following fermionic matrix element (cf. [15])

$$\phi^{(n)}_{\lambda\mu}(t) = W_{\mu\to\lambda}(t,n) = \langle\lambda;n|e^{t\sum_{i=1}^{\infty} r_i\psi_i\psi^\dagger_{i-1}}|\mu;n\rangle, \tag{4.43}$$

which satisfies

$$\frac{d\phi^{(n)}_{\lambda\mu}(t)}{dt} = \sum_{\nu} M_{\lambda\nu}M_{\nu\mu}, \tag{4.44}$$

where $\{r_i\}_{i\in\mathbf{N}}$ are positive real constants giving the infinitesimal transition rates and

$$M_{\nu\mu} = \langle\nu;n|\sum_{i=1}^{\infty} r_i\psi_i\psi^\dagger_{i-1}|\mu;n\rangle. \tag{4.45}$$

The choice (4.45) corresponds to a polynomial system $\{\phi_i\}_{i\in\mathbf{N}}$ with coefficient matrix

$$\phi = e^{\alpha}, \tag{4.46}$$

where

$$\alpha = \begin{pmatrix} \ddots & \ddots & \vdots & \cdots & \vdots \\ \ddots & r_{j+1} & 0 & \cdots & 0 \\ \cdots & 0 & r_j & \ddots & \vdots \\ \cdots & 0 & 0 & \ddots & 0 \\ \cdots & \vdots & \vdots & \ddots & r_1 \\ \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.47}$$

The associated τ-function (3.44) for this case

$$\tau_\phi(n,\mathbf{s},\mathbf{t}) = \sum_{\lambda,\mu} W_{\mu\to\lambda}(t,n)S_\lambda(\mathbf{s})S_\mu(\mathbf{t}), \tag{4.48}$$

may therefore be viewed as a generating function for the transition probabilities .

Note however that (4.45) is not a normalized transition probability, since the Markov condition

$$\sum_{\nu} M_{\nu\mu} = 0 \tag{4.49}$$

is not satisfied. To correct this, it would be necessary to add a further term that is quartic in the ψ's and $\psi^\dagger$'s. But the generating function would then no longer be a τ-function; i.e., this would not longer be a "free fermion" process. To obtain a normalized transition probability in (4.43), it is therefore necessary to renormalize continuously at all time values, by dividing by the sum of transition rates to all final states. Alternatively, the exponential factor in (4.43) can be developed as a power series in t, with the powers viewed as discrete times. The expansion may then be viewed as a generating function for a discrete time exclusion process, as in [15], and the normalization by division of the relative weights by the sum over all final states can be applied at each discrete time value.

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Acknowledgements. The authors would like to thank A. Yu. Orlov for helpful comments and assistance with the proof of Proposition 3.1, A. Borodin and G. Olshanski for pointing out ref. [16] and A. Veselov for helping to clarify example 4.1.

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